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Iterative Methods for Generalized Variational Inequalities

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Abstract—In this paper, we use the auxiliary principle technique to suggest a new class of predictor-corrector algorithms for solving generalized variational inequalities. The convergence of the proposed method only requires the partially relaxed strongly monotonicity of the operator, which is weaker than co-coercivity. As special cases, we obtain a number of known and new results for solving various classes of variational inequalities.

Keywords—Variational inequalities, Auxiliary principle, Iterative methods, Convergence.

1. INTRODUCTION

Variational inequality theory is a branch of applicable mathematical sciences with a wide range of applications in industry, physical, pure and applied sciences, see [1–18]. Variational inequalities have been generalized and extended in various directions using novel and innovative techniques. A useful and important generalization of the variational inequalities is the generalized variational inequality, which is mainly due to Fang and Peterson [3]. There are several numerical methods for solving variational inequalities. In recent years, a number of three-step forward-backward splitting type methods have been suggested by using the updating technique of the solution and the auxiliary principle approach for solving the classical variational inequalities. It is known that the technique of the updating solution cannot be used to suggested similar splitting methods for solving generalized variational inequalities. In this paper, we use the auxiliary principle technique to suggest and analyze a class of predictor-corrector methods for solving generalized variational inequalities. We remark that a number of iterative methods including the projection and its variant forms, forward-backward splitting technique can be obtained from these predictor-corrector methods as special cases. We show that the convergence of these new methods requires only the partially relaxed strongly monotonicity, which is a weaker condition than the co-coercivity. This shows that our results improve and refine the known convergence results of the iterative methods.

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2. PREMILINARIES

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|.\|$, respectively. Let $C(H)$ be a family of all nonempty compact subset of H. Let $T : H \longrightarrow C(H)$ be a multivalued operator. Let K be a nonempty closed convex set in H.

We consider the problem of finding $u \in K$, $\nu \in T(u)$ such that

$$\langle \nu, v-u\rangle \geq 0, \qquad \text{for all } v \in K. \tag{2.1}$$

The inequality of type (2.1) is called the *generalized variational inequality*, introduced and studied by Fang and Peterson [3]. It can be shown that a wide class of multivalued free, obstacle, moving, equilibrium, and optimization problems arising in pure and applied sciences can be studied via the generalized variational inequalities (2.1), see [3,10].

We note that if $T : H \longrightarrow H$ is a single-valued operator, then problem (2.1) is equivalent to finding $u \in K$ such that

$$\langle Tu, v-u\rangle \geq 0, \qquad \text{for all } v \in K, \tag{2.2}$$

which is known as the classical variational inequality, introduced and studied by Stampacchia [17] in 1964. For the recent applications, numerical methods, and sensitivity analysis of the variational inequalities, see [2–18] and the references therein.

We also recall the following known result and concepts.

LEMMA 2.1. *For all $u, v \in H$, we have*

$$\langle u, v\rangle = \frac{1}{2}\left\{\|u+v\|^2 - \|u\|^2 - \|v\|^2\right\}.$$

DEFINITION 2.1. *For all u_1, u_2, $z \in H$, $w_1 \in T(u_1)$, $w_2 \in T(u_2)$, the multivalued operator $T : H \to C(H)$ is said to be*

(i) *partially relaxed strongly monotone, if there exists a constant $\alpha > 0$ such that*

$$\langle w_1 - w_2, z - u_2\rangle \geq -\alpha\|u_1 - z\|^2,$$

(ii) *co-coercive, if there exists a constant $\mu > 0$ such that*

$$\langle w_1 - w_2, u_1 - u_2\rangle \geq \mu\|w_1 - w_2\|^2,$$

(iii) *M-Lipschitz continuous, if there exists a constant $\delta > 0$ such that*

$$M\left(T(u_1), T(u_2)\right) \leq \delta\|u_1 - u_2\|,$$

where $M(.,.)$ is the Hausdorff metric on $C(H)$. We remark that if $z = u_1$, then partially relaxed strongly monotonicity is exactly monotonicity of the operator T. For the single-valued operator T, Definition 2.1 reduces to the definition of partially relaxed strongly monotonicity and co-coercivity of the operator. It is known that the co-coercivity implies the partially relaxed strongly monotonicity, but the converse is not true.

3. MAIN RESULTS

In this section, we suggest and analyze a new iterative method for solving problem (2.1) by using the auxiliary principle technique.

For a given $u \in K$, consider the problem of finding a unique $w \in K$, $\eta \in T(w)$ satisfying the auxiliary variational inequality

$$\langle \rho\eta + w - u, v - w\rangle \geq 0, \qquad \text{for all } v \in K, \tag{3.1}$$

where $\rho > 0$ is a constant.

We note that if $w = u$, then clearly w is a solution of the generalized variational inequality (2.1). This observation enables us to suggest the following predictor-corrector method for solving the generalized variational inequalities (2.1).

ALGORITHM 3.1. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes

$$\langle \rho\eta_n + u_{n+1} - w_n, v - u_{n+1} \rangle \geq 0, \qquad \text{for all } v \in K, \tag{3.2}$$

$$\eta_n \in T(w_n) : \|\eta_{n+1} - \eta_n\| \leq M\left(T\left(w_{n+1}\right), T\left(w_n\right)\right), \tag{3.3}$$

$$\langle \beta\xi_n + w_n - y_n, v - w_n \rangle \geq 0, \qquad \text{for all } v \in K, \tag{3.4}$$

$$\xi_n \in T\left(y_n\right) : \|\xi_{n+1} - \xi_n\| \leq M\left(T\left(y_{n+1}\right), T\left(y_n\right)\right), \tag{3.5}$$

and

$$\langle \mu\nu_n + y_n - u_n, v - y_n \rangle \geq 0, \qquad \text{for all } v \in K, \tag{3.6}$$

$$\nu_n \in T(u_n) : \|\nu_{n+1} - \nu_n\| \leq M(T(u_{n+1}), T(u_n)), \qquad n = 0, 1, 2, \dots, \tag{3.7}$$

where $\rho > 0$, $\mu > 0$, and $\beta > 0$ are constants.

Using the technique of the projection, Algorithm 3.1 can be written as follows.

ALGORITHM 3.2. For a given $u_0 \in H$, compute u_{n+1} such that $\eta_n \in T(w_n)$, $\xi_n \in T(y_n)$, $\nu_n \in T(u_n)$ by the iterative schemes

$$\begin{aligned} u_{n+1} &= P_K\left[w_n - \rho\eta_n\right], \\ w_n &= P_K\left[y_n - \beta\xi_n\right], \\ y_n &= P_K\left[u_n - \mu\nu_n\right], \qquad n = 0, 1, 2 \dots. \end{aligned}$$

Algorithm 3.2 is a three-step forward-backward splitting method for solving generalized variational inequalities (2.1), which appears to be a new one.

If T is a single-valued operator, then Algorithm 3.1 collapses to the following predictor-corrector method for solving variational inequalities (2.2), which is due to Noor [13].

ALGORITHM 3.3. For a given $u_0 \in H$, compute u_{n+1} by the iterative schemes

$$\begin{aligned} \langle \rho T\left(w_n\right) + u_{n+1} - w_n, v - u_{n+1} \rangle &\geq 0, \qquad \text{for all } v \in K, \\ \langle \beta T\left(y_n\right) + w_n - y_n, v - w_n \rangle &\geq 0, \qquad \text{for all } v \in K, \\ \langle \mu T\left(u_n\right) + y_n - u_n, v - y_n \rangle &\geq 0, \qquad \text{for all } v \in K. \end{aligned}$$

Using the projection technique, Algorithm 3.3 can be written in the equivalent form as follows.

ALGORITHM 3.4. For a given $u_0 \in H$, compute u_{n+1} by the iterative schemes

$$\begin{aligned} y_n &= P_K\left[u_n - \mu T\left(u_n\right)\right], \\ w_n &= P_K\left[y_n - \beta T\left(y_n\right)\right], \\ u_{n+1} &= P_K\left[w_n - \rho T\left(w_n\right)\right], \qquad n = 0, 1, 2 \dots, \end{aligned}$$

which can be written in the following form:

$$u_{n+1} = P_K\left[I - \rho T\right] P_K[I - \beta T] P_K[I - \mu T] u_n, \qquad n = 0, 1, 2 \dots,$$

which is a three-step forward-backward splitting algorithm. Algorithm 3.4 is similar to the so-called θ-scheme of Glowinski and Le Tallec [5], which they suggested by using the Lagrangian multiplier method. It has been shown in [5] that three-step schemes are numerically efficient. For the applications of the splitting methods in partial differential equations, see [19] and the references therein. The convergence analysis of Algorithm 3.4 has been considered by Noor [13].

For the convergence analysis of Algorithm 3.1, we need the following result.

LEMMA 3.1. *Let $u \in H$ be the exact solution of (2.1) and u_{n+1} be the approximate solution obtained from Algorithm 3.1. If the operator $T : H \longrightarrow C(H)$ is a partially relaxed strongly monotone operator with constant $\alpha > 0$, then*

$$\|u_{n+1} - u\|^2 \leq \|u_n - u\|^2 - (1 - 2\rho\alpha) \|u_{n+1} - u_n\|^2. \tag{3.8}$$

PROOF. Let $u \in K$, $\nu \in T(u)$ be a solution of (2.1). Then

$$\langle \rho\nu, v - u\rangle \geq 0, \qquad \text{for all } v \in K, \tag{3.9}$$

$$\langle \beta\nu, v - u\rangle \geq 0, \qquad \text{for all } v \in K, \tag{3.10}$$

$$\langle \mu\nu, v - u\rangle \geq 0, \qquad \text{for all } v \in K, \tag{3.11}$$

where $\rho > 0$, $\beta > 0$, and $\mu > 0$ are constants.

Now taking $v = u_{n+1}$ in (3.9) and $v = u$ in (3.2), we have

$$\langle \rho\nu, u_{n+1} - u\rangle \geq 0 \tag{3.12}$$

and

$$\langle \rho\eta_n + u_{n+1} - w_n, u - u_{n+1}\rangle \geq 0. \tag{3.13}$$

Adding (3.12) and (3.13), we have

$$\langle u_{n+1} - w_n, u - u_{n+1}\rangle \geq \rho \langle \eta_n - \nu, u_{n+1} - u\rangle \geq -\alpha\rho \|u_{n+1} - w_n\|^2, \tag{3.14}$$

where we have used the fact that T is partially relaxed strongly monotone with constant $\alpha > 0$.

Setting $u = u - u_{n+1}$ and $v = u_{n+1} - w_n$ in (2.3), we obtain

$$\langle u_{n+1} - w_n, u - u_{n+1}\rangle = \frac{1}{2}\left\{\|u - w_n\|^2 - \|u - u_{n+1}\|^2 - \|u_{n+1} - w_n\|^2\right\}. \tag{3.15}$$

Combining (3.14) and (3.15), we have

$$\|u_{n+1} - u\|^2 \leq \|w_n - u\|^2 - (1 - 2\alpha\rho) \|u_{n+1} - w_n\|^2. \tag{3.16}$$

Taking $v = u$ in (3.3) and $v = w_n$ in (3.10), we have

$$\langle \beta\nu, w_n - u\rangle \geq 0 \tag{3.17}$$

and

$$\langle \beta\xi_n + w_n - y_n, u - w_n\rangle \geq 0. \tag{3.18}$$

Adding (3.17) and (3.18) and rearranging the terms, we have

$$\langle w_n - y_n, u - w_n\rangle \geq \beta \langle \xi_n - \nu, w_n - u\rangle \geq -\beta\alpha \|y_n - w_n\|^2, \tag{3.19}$$

since T is a partially relaxed strongly monotone operator with constant $\alpha > 0$.

Now taking $v = w_n - y_n$ and $u = u - w_n$ in (2.3), equation (3.19) can be written as

$$\|u - w_n\|^2 \leq \|u - y_n\|^2 - (1 - 2\beta\alpha) \|y_n - w_n\|^2 \leq \|u - y_n\|^2, \qquad \text{for } 0 < \beta < \frac{1}{2}\alpha. \tag{3.20}$$

Similarly, by taking $v = u$ in (3.4) and $v = u_{n+1}$ in (3.11) and using the partially relaxed strongly monotonicity of the operator T, we have

$$\langle y_n - u_n, u - y_n \rangle \geq -\mu\alpha \|y_n - u_n\|^2. \tag{3.21}$$

Letting $v = y_n - u_n$, and $u = u - y_n$ in (2.3), and combining the resultant with (3.21), we have

$$\|y_n - u\|^2 \leq \|u - u_n\|^2 - (1 - 2\mu\alpha)\|y_n - u_n\|^2 \leq \|u - u_n\|^2, \qquad \text{for } 0 < \mu < \frac{1}{2\alpha}. \tag{3.22}$$

Now

$$\begin{aligned} \|u_{n+1} - w_n\|^2 &= \|u_{n+1} - u_n + u_n - w_n\|^2 \\ &= \|u_{n+1} - u_n\|^2 + \|u_n - w_n\|^2 + 2\langle u_{n+1} - u_n, u_n - w_n \rangle. \end{aligned} \tag{3.23}$$

Combining (3.16), (3.20), (3.22), and (3.23), we obtain

$$\|u_{n+1} - u\|^2 \leq \|u_n - u\|^2 - (1 - 2\beta\alpha)\|u_{n+1} - u_n\|^2,$$

the required result (3.8). ∎

THEOREM 3.1. *Let H be a finite-dimensional space and $0 < \rho < 1/2\alpha$. Let $T : H \longrightarrow C(H)$ be an M-Lipschitz continuous operator. If u_{n+1} is the approximate solution obtained from Algorithm 3.1 and $u \in H$ is the exact solution of (2.1), then $\lim_{n\to\infty} u_n = u$.*

PROOF. Let $u \in H$ be a solution of (2.1). Since $0 < \rho < 1/2\alpha$. From (3.8), it follows that the sequence $\{\|u - u_n\|\}$ is nonincreasing, and consequently, $\{u_n\}$ is bounded. Furthermore, we have

$$\sum_{n=0}^{\infty}(1 - 2\alpha\rho)\|u_{n+1} - u_n\|^2 \leq \|u_0 - u\|^2,$$

which implies that

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \tag{3.24}$$

Let $\hat{u}$ be the cluster point of $\{u_n\}$ and let the subsequence $\{u_{n_j}\}$ of the sequence $\{u_n\}$ converge to $\hat{u} \in H$. Replacing w_n and y_n by u_{n_j} in (3.2), (3.4), and (3.6), taking the limit $n_j \longrightarrow \infty$ and using (3.24), we have

$$\langle \hat{\nu}, v - \hat{u} \rangle \geq 0, \qquad \text{for all } v \in K,$$

which implies that $\hat{u}$ solves the generalized variational inequality (2.1) and

$$\|u_{n+1} - u\|^2 \leq \|u_n - u\|^2.$$

Thus, it follows from the above inequality that the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$ and

$$\lim_{n\to\infty} u_n = \hat{u}.$$

It remains to show that $\nu \in T(u)$. From (3.7) and using the M-Lipschitz continuity of the multivalued operator T, we have

$$\|\nu_n - \nu\| \leq M(T(u_n), T(u)) \leq \delta\|u_n - u\|,$$

which implies that $\nu_n \longrightarrow \nu$ as $n \longrightarrow \infty$. Now consider

$$\begin{aligned} d(\nu, T(u)) &\leq \|\nu - \nu_n\| + d(\nu, T(u)) \\ &\leq \|\nu - \nu_n\| + M(T(u_n), T(u)) \\ &\leq \|\nu - \nu_n\| + \delta\|u_n - u\| \longrightarrow 0, \qquad \text{as } n \longrightarrow \infty, \end{aligned}$$

where $d(\nu, T(u)) = \inf\{\|\nu - z\| : z \in T(u)\}$ and $\delta > 0$ is the M-Lipschitz continuity constant of the operator T. From the above inequality, it follows that $d(\nu, T(u)) = 0$. This implies that $\nu \in T(u)$, since $T(u) \in C(H)$. This completes the proof. ∎

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